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Numerical Exploration of a Family of Strictly Convex Billiards With Boundary of Class C^2

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We are interested in the possible existence of strictly convex ergodic billiards. Such billiards are searched for by means of numerical investigation. The boundary of a billiard is built with four arcs of class C^{∞}. Adjacent arcs have equal curvatures at connecting points. The surface of section of the billiards is explored. It seems as if symmetric billiards always have invariant curves (islands). Asymmetric billiards have been found which look ergodic. They are built with an arc of an ellipse, two arcs of circles, and one-half of a Descartes oval.

KEY WORDS: Dynamical systems; classical billiards; ergodic billiards; numerical exploration.

1. INTRODUCTION

Let Ω be an open region of $\mathbf{R}^2$ whose boundary is a strictly convex oriented closed curve $\partial\Omega$ of class C^k, $k \geqslant 2$. A billiard in Ω is the dynamical system defined by either the free motion of a particle or a light ray in the interior of this enclosure with specular bounces or reflections on the boundary.

A convenient way to explore the properties of a billiard is through the surface of section. We look at successive reflections. Let η be the length, measured along $\partial\Omega$, from the point of impact to an arbitrarily chosen origin, and θ the oriented angle measured from the normal to $\partial\Omega$ at the point of impact to the incident ray. We suppose that $\partial\Omega$ has unit length. We take as coordinates in the surface of section η and $S = \sin\theta$. It is clear

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that $0 \leqslant \eta \leqslant 1$ and $|S| < 1$. The surface of section is then identified with the rectangle

$$\Pi = \{(\eta, S) \in \mathbf{R}^2;\ 0 \leqslant \eta \leqslant 1,\ -1 < S < 1\}$$

The trajectory in the billiard is represented by a map $\Phi: \Pi \to \Pi$ that preserves the Lebesgue measure $d\eta\, dS$.

2. CAUSTICS AND INVARIANT CURVES

A closed convex curve $\mathscr{C}$ inside $\partial\Omega$ is said to be a caustic of $\partial\Omega$ if it is such that a ray that leaves $\partial\Omega$ being tangent to $\mathscr{C}$ remains so after reflection on $\partial\Omega$. The caustic is oriented like $\partial\Omega$.

It is known[1-3] that if the boundary of the billiard is sufficiently smooth ($k \geqslant 6$) there is a discontinuous family of caustics in a small neighborhood of $\partial\Omega$ and the union of caustics has positive measure. To every caustic of $\partial\Omega$ of class C^ν, $\nu \geqslant 2$, Φ associates an invariant curve of class $C^{\nu-1}$ in the surface of section; it is the graph of the mapping which associates to η the sine of the oriented angle between the normal at the point of coordinate η and the tangent to $\mathscr{C}$ drawn from that point.

Invariant curves in the surface of section may either run from edge $\eta = 0$ to edge $\eta = 1$ (that is the case for those invariant curves associated with caustics close to the border of the billiard) or have the shape of islands surrounding invariant points which correspond to stable periodic orbits in the billiard.

It is well known that if $\partial\Omega$ is an ellipse, the surface of section is filled with such curves and the billiard is said to be integrable.

We say that a billiard is ergodic if there are no invariant curves the union of which has a positive measure in the surface of section.

3. A FAMILY OF BILLIARDS OF CLASS C^2

According to refs. 1–3, it is clear that a strictly convex billiard with sufficiently smooth boundary cannot be ergodic. Is this still the case when the boundary is not so smooth?

Numerical exploration of several families of strictly convex asymmetric C^1 billiards built with four arcs of circles[4] has shown that there are billiards belonging to these families that are candidates for ergodicity.

By the same method we have searched for billiards with C^2 boundary that may be ergodic. We started by exploring billiards with either one or two axes of symmetry constructed with four arcs of circles or ellipses. We found that such billiards always have islands in their surface of section.

This seems to be the case also for more sophisticated billiards with only one axis of symmetry, for instance, billiards constructed with two arcs of a circle, one arc of an ellipse, and half of a Cassini oval. It seems as if symmetric billiards (with either one or two axes of symmetry) always have elliptic invariant points in the surface of section which correspond either to a periodic orbit along the minor diameter or to a multiple periodic orbit that lies close the minor diameter. See refs. 4 and 5 for examples of such orbits.

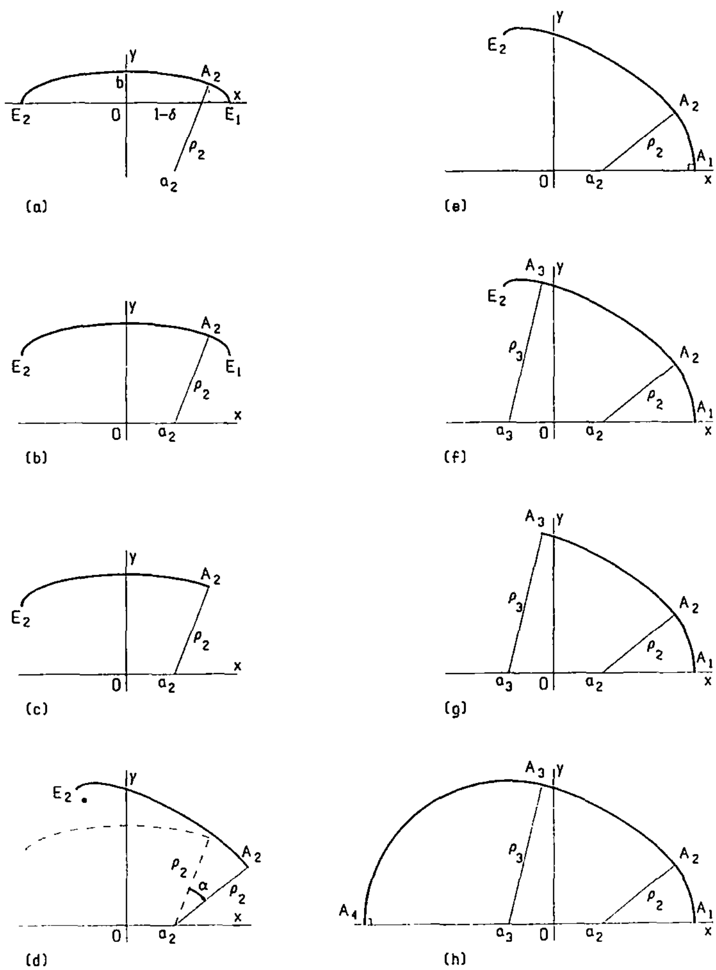

Fig. 1. The construction of the upper part of the billiard.

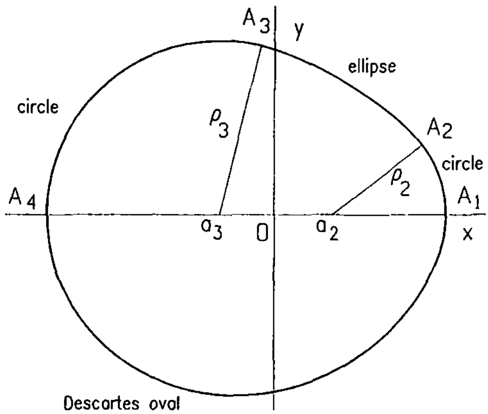

Fig. 2. The general shape of a billiard.

. This is the reason why we investigated asymmetric billiards. Figures 1a–1h and 2 describe the construction of a family of such C^2 billiards. To construct a billiard of the family, proceed as follows.

(a) On the upper half $(E_1 E_2)$ of the ellipse of semimajor axis 1 and semiminor axis b take a point A_2 with abcissa $1-\delta$ $(\delta > 0)$. Find the center of curvature a_2 of the ellipse at point A_2. The corresponding radius of curvature is ρ_2.

(b) Shift the figure parallel to the y axis so as to bring a_2 on the x axis.

(c) Drop arc $A_2 E_1$

(d) Rotate arc $E_2 A_2$ around a_2 by an angle $-\alpha(\alpha > 0)$; α has to be smaller than some $\alpha_{\max}$ (which would bring A_2 on the x axis) in order for the construction to proceed.

(e) Construct the arc of circle $A_2 A_1$ with center a_2 and radius ρ_2. Point A_1 is on the x axis.

(f) Find the point A_3 on the arc of ellipse $A_2 E_2$ such that the center of curvature a_3 at A_3 is on the x axis. The corresponding radius of curvature is ρ_3.

(g) Drop arc $E_2 A_3$.

(h) Construct the arc of circle $A_3 A_4$ with center a_3 and radius ρ_3. Point A_4 is on the x axis

Construct as in Fig. 2 the lower half Descartes oval $(y < 0)$ passing through A_4 and A_1 and such that the radius of curvature of the oval at A_4

(resp. A_3) is ρ_3 (resp. ρ_2). Let us recall that a Descartes oval is the locus of the points M of the plane such that given two fixed points F and F', one has $MF + kMF' = 2a$, where $k > 0$, $k \neq 1$. A Descartes oval is an algebraic curve of degree 4. If the length of the major diameter and the radii of curvature at the ends of the major diameter of the oval are given, the oval is unique.

4. NUMERICAL EXPLORATION OF THE BILLIARDS

We have defined in this way a three-parameter family of billiards. The parameters are b, δ, and α, which will be given in degrees.

The natural coordinates in the surface of section are η, S (A_1 is taken as origin for the η's). As η and S are canonically conjugate, the corresponding Poincaré map is area preserving. However, it is more convenient to use the angle $\phi = A_1M$ instead of η; see Fig. 3 (ϕ is a continuous function of η). Although the map of the surface of section onto itself does not preserve the measure $d\phi\, dS$, the topology of invariant curves is unchanged.

Figures 4a–4f show the surfaces of section for billiards with $b = 0.09$, $\delta = 0.05$, and several values of α. For these values of b and δ one has $\alpha_{max} = 74.686\ldots$.

Computations have been performed in double precision; 10^6–10^7 bounces have been computed for each billiard. The representation of the surface of section is discretized in a square containing 512×512 cells, as in ref. 4.

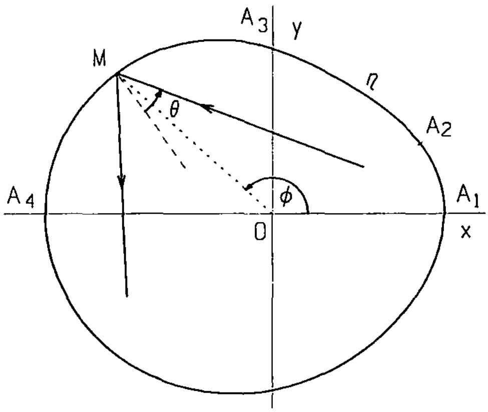

Fig. 3. The geometry of a bounce.

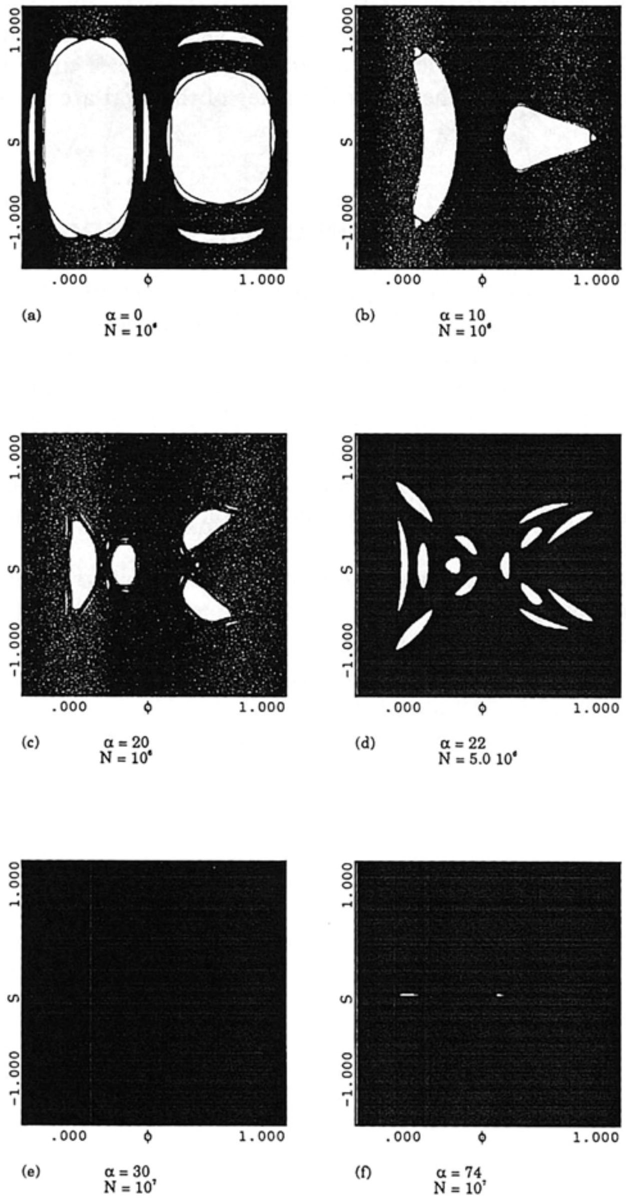

Fig. 4. The surface of section of six billiards of the family $b = 0.09$, $\delta = 0.05$. The billiard with $\alpha = 0$ has one axis of symmetry. All other billiards are asymmetric. The billiard with $\alpha = 30$ looks ergodic.

When $\alpha = 0$ the billiard is symmetric with respect to the y axis. It is asymmetric when $\alpha \neq 0$. The periodic orbit along the major diameter of the billiard is always unstable.[6] The periodic orbit along the minor diameter is stable for billiards with $0 \leqslant \alpha < 12.05...$; Fig. 4a shows the surface of section of the billiard $\alpha = 0$. The large central islands encircle two invariant points which correspond to the stable periodic orbit along the minor diameter of the billiard. The islands are filled with invariant curves (not shown on the figure). Figure 4b is for the billiard $\alpha = 10$. The periodic orbit along the minor diameter is again stable. This orbit is unstable for billiards with $12.05... < \alpha < 42.87...$ and again stable for billiards with $42.87... < \alpha < \alpha_{max}$. For $\alpha = 12.05...$ one observes the first of a series of bifurcations with period doubling of the orbit along the minor diameter of the billiard. Figure 4c is for the billiard $\alpha = 20$. The four central islands are around the invariant points which correspond to a 2-periodic orbit that lies close to the minor diameter of the billiard and starts normally to the boundary ($S = 0$). The islands are filled with invariant curves (not shown). Figure 4d is for the billiard $\alpha = 22$. It shows islands around the 14 invariant points associated with a periodic orbit which does not belong to the cascade mentioned above.

A systematic exploration conducted by giving to α all permissible values show that billiards look ergodic in a whole range of values of α beyond the limits of period doubling. The billiard with $\alpha = 30$ is an example. However, this result cannot be taken for granted even from the numerical exploration point of view, as it is known that islets of stability can be found beyond period-doubling limits. This is shown, for instance, in ref. 7. Such very small islands could be undetectable at the scale of our figures (and even at a larger scale).

5. THEORETICAL ASPECTS

The suggestion that the above billiard with $\alpha = 30$ is ergodic may seem in contradiction with a conjecture stated by Bunimovich.[8] It is known that the mechanism of strong defocusing is the only one that produces chaos in billiards with focusing components of the boundary. Bunimovich makes the conjecture that if a billiard has a regular focusing component which *is not* absolutely focusing, then the billiard has a stable periodic orbit. (See ref. 8 for the definitions of focusing and absolutely focusing components and the mechanism of defocusing.)

A piece of one of the regular components of the above billiard with $\alpha = 30$ is part of the half-ellipse with an axis ratio of $100/9 > \sqrt{2}$. The half-ellipse *is not* absolutely focusing,[9] but it is not clear if the same is true for the arc that has been taken in the construction of the billiard. We could not determine if the half oval is not absolutely focusing.

We tried nevertheless to find experimentally a stable periodic orbit along the lines suggested in ref. 8. The method consists in looking for parallel beams of rays that become almost parallel, but converge after a series of consecutive reflections from the elliptical component or from the oval component.

We could not find such beams. We do not claim they do not exist, because we explored only a fraction of all possible situations. However, this negative result is not very surprising, because Fig. 4e suggests that were a stable periodic orbit to exist, the region of stability islands would be very small.

6. CONCLUSIONS

If billiards of the C^2 family described above do not have caustics along the border, numerical exploration seems to suggest that some of them could be ergodic. This may be in contradiction with a conjecture by Bunimovich on billiards with components which are not absolutely focusing. If the billiards are not ergodic, stability islands in the surface of section are extremely small.

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REFERENCES

1. V. F. Lazutkin, The existence of caustics for a billiard problem in a convex domain, *Math. USSR Isz.* 7:185–214 (1973).
2. V. F. Lazutkin, *Convex billiards and Eigenfunctions of Laplace Operator*, (University of Leningrad Editions, 1981), p. 196 [in Russian].
3. R. Douady, Applications du théorème des tores invariants, Thèse de 3ème cycle, Université Paris VII (1982).
4. A. Hayli and T. Dumont, Expériences numériques sur des billards C^1 formés de quatre arcs de cercles, *Cel. Mech.* **38**:23–66 (1986).
5. A. Hayli, T. Dumont, J. Moulin-Ollagnier, and J.-M. Strelcyn, Quelques résultats nouveaux sur les billards de Robnik, *J. Phys. A. Math. Gen.* **20**:3237–3249 (1987).
6. G. D. Birkhoff, *Dynamical Systems* (American Mathematical Society, Providence, Rhode Island, 1927), p. 169.

7. R. S. MacKay, Islets of stability beyond period doubling, *Phys. lett.* **87A**:321–324 (1982).
8. L. A. Bunimovich, Conditions of stochasticity of two-dimensional billiards, *Chaos* **1**:187–193 (1991).
9. V. J. Donnay, Using integrability to produce chaos: Billiards with positive entropy, *Commun. Math. Phys.* **141**:225–257 (1991).